

# A renormalisation-group treatment of two-body scattering

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**Abstract.** A Wilsonian renormalisation group is used to study nonrelativistic two-body scattering by a short-ranged potential. We identify two fixed points: a trivial one and one describing systems with a bound state at zero energy. The eigenvalues of the linearised renormalisation group are used to assign a systematic power-counting to terms in the potential near each of these fixed points. The expansion around the nontrivial fixed point is shown to be equivalent to the effective-range expansion.

## INTRODUCTION

Recently there has been much interest in the possibility of developing a systematic treatment of low-energy nucleon-nucleon scattering using the techniques of effective field theory [1–3]. Here we approach the problem using Wilson’s continuous renormalisation group [4] to examine the low-energy scattering of nonrelativistic particles interacting through short-range forces [5].

The starting point for the renormalisation group (RG) is the imposition of a momentum cut-off,  $|\mathbf{k}| < \Lambda$ , separating the low-momentum physics which we are interested in from the high-momentum physics which we wish to “integrate out”. Provided that there is a separation of scales between these two regimes, we may demand that low-momentum physics should be independent of  $\Lambda$ .

The second step is to rescale the theory, expressing all dimensioned quantities in units of  $\Lambda$ . As the cut-off  $\Lambda$  approaches zero, all physics is integrated out until only  $\Lambda$  itself is left to set the scale. In units of  $\Lambda$  any couplings that survive are just numbers, and these define a “fixed point”. Such fixed points correspond to systems with no natural momentum scale. Examples include the trivial case of a zero scattering amplitude and the more interesting one of a bound state at exactly zero energy.

Real systems can then be described in terms of perturbations away from one of these fixed points. For perturbations that scale as definite powers of  $\Lambda$ , we can set up a power-counting scheme: a systematic way to organise the terms in an effective potential or an effective field theory. A fixed point is said to be stable if

all perturbations vanish like positive powers of  $\Lambda$  as  $\Lambda \rightarrow 0$  and unstable if one or more of them grows with a negative power of  $\Lambda$ .

## TWO-BODY SCATTERING

We consider  $s$ -wave scattering by a potential that consists of contact interactions only. Expanded in powers of energy and momentum this has the form

$$V(k', k, p) = C_{00} + C_{20}(k^2 + k'^2) + C_{02}p^2 \dots, \quad (1)$$

where  $k$  and  $k'$  denote momenta and energy-dependence is expressed in terms of the on-shell momentum  $p = \sqrt{ME}$ . Below all thresholds for production of other particles, this potential should be an analytic function of  $k^2$ ,  $k'^2$  and  $p^2$ .

Low-energy scattering is conveniently described in terms of the reactance matrix,  $K$ . This is similar to the scattering matrix  $T$ , except for the use of standing-wave boundary conditions. It satisfies the Lippmann-Schwinger (LS) equation (see [6])

$$K(k', k, p) = V(k', k, p) + \frac{M}{2\pi^2} \mathcal{P} \int q^2 dq \frac{V(k', q, p)K(q, k, p)}{p^2 - q^2}, \quad (2)$$

where  $\mathcal{P}$  denotes the principal value.

On-shell, with  $k = k' = p$ , the  $K$ -matrix is related to the phase-shift by

$$\frac{1}{K(p, p, p)} = -\frac{M}{4\pi} p \cot \delta(p), \quad (3)$$

which means it has a simple relation to the effective-range expansion [7],

$$p \cot \delta(p) - \frac{1}{a} + \frac{1}{2} r_e p^2 + \dots, \quad (4)$$

where  $a$  is the scattering length and  $r_e$  is the effective range. We shall see that this turns out to be equivalent to an expansion around a nontrivial fixed point of the RG.

## RENORMALISATION GROUP

To set up the RG we first impose a momentum cut-off on the intermediate states in the LS equation (2). This can be written

$$K = V(\Lambda) + V(\Lambda)G_0(\Lambda)K, \quad (5)$$

where we have included a sharp cut-off in the free Green's function,

$$G_0 = \frac{M\theta(\Lambda - q)}{p^2 - q^2}. \quad (6)$$

We now demand that  $V(k', k, p, \Lambda)$  varies with  $\Lambda$  in order to keep the off-shell  $K$ -matrix independent of  $\Lambda$ :

$$\frac{\partial K}{\partial \Lambda} = 0. \quad (7)$$

This is sufficient to ensure that all scattering observables do not depend on  $\Lambda$ . Differentiating the LS equation (5) with respect to  $\Lambda$  and then operating from the right with  $(1 + G_0 K)^{-1}$ , we get

$$\frac{\partial V}{\partial \Lambda} = \frac{M}{2\pi^2} V(k', \Lambda, p, \Lambda) \frac{\Lambda^2}{\Lambda^2 - p^2} V(\Lambda, k, p, \Lambda). \quad (8)$$

We now introduce dimensionless momentum variables,  $\hat{k} = k/\Lambda$  etc., and a rescaled potential,

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = \frac{M\Lambda}{2\pi^2} V(\Lambda \hat{k}', \Lambda \hat{k}, \Lambda \hat{p}, \Lambda). \quad (9)$$

From the equation (8) satisfied by  $V$  we find that the rescaled potential satisfies the RG equation

$$\Lambda \frac{\partial \hat{V}}{\partial \Lambda} = \hat{k}' \frac{\partial \hat{V}}{\partial \hat{k}'} + \hat{k} \frac{\partial \hat{V}}{\partial \hat{k}} + \hat{p} \frac{\partial \hat{V}}{\partial \hat{p}} + \hat{V} + \hat{V}(\hat{k}', 1, \hat{p}, \Lambda) \frac{1}{1 - \hat{p}^2} \hat{V}(1, \hat{k}, \hat{p}, \Lambda). \quad (10)$$

## FIXED POINTS

We are now in a position to look for fixed points: solutions of (10) that are independent of  $\Lambda$ . These provide the possible low-energy limits of theories as  $\Lambda \rightarrow 0$  and hence the starting points for systematic expansions of the potential.

### The trivial fixed point

One obvious solution of (10) is the trivial fixed point,

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = 0, \quad (11)$$

which describes a system with no scattering.

For systems described by potentials close to the fixed point we can expand in terms of eigenfunctions,  $\hat{V} = \Lambda^\nu \phi(\hat{k}', \hat{k}, \hat{p})$ , of the linearised RG equation,

$$\hat{k}' \frac{\partial \phi}{\partial \hat{k}'} + \hat{k} \frac{\partial \phi}{\partial \hat{k}} + \hat{p} \frac{\partial \phi}{\partial \hat{p}} + \phi = \nu \phi. \quad (12)$$

These have the form

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = C\Lambda^\nu \hat{k}'^l \hat{k}^m \hat{p}^n, \quad (13)$$

with eigenvalues  $\nu = l + m + n + 1$ , where  $l, m$  and  $n$  are non-negative even integers. The eigenvalues are all positive and so the fixed point is a stable one: all nearby potentials flow towards it as  $\Lambda \rightarrow 0$ .

The corresponding unscaled potential has the expansion

$$V(\hat{k}', \hat{k}, \hat{p}, \Lambda) = \frac{2\pi^2}{M} \sum_{l,n,m} \hat{C}_{lmn} \Lambda_0^{-\nu} k'^l k^m p^n, \quad (14)$$

where we have written the coefficients in dimensionless form by taking out powers of  $\Lambda_0$ , the scale of the short-distance physics. The power counting in this expansion is just the one proposed by Weinberg [1] if we assign an order  $d = \nu - 1$  to each term in the potential. This fixed point can be used to describe systems where the scattering at low energies is weak and can be treated perturbatively. It is not the appropriate starting point for  $s$ -wave nucleon-nucleon scattering, where the scattering length is large.

## A nontrivial fixed point

The simplest nontrivial fixed point is one that depends on energy only,  $\hat{V} = \hat{V}_0(\hat{p})$ . It satisfies

$$\hat{p} \frac{\partial \hat{V}_0}{\partial \hat{p}} + \hat{V}_0(\hat{p}) + \frac{\hat{V}_0(\hat{p})^2}{1 - \hat{p}^2} = 0. \quad (15)$$

The solution, which must be analytic in  $\hat{p}^2$ , is

$$\hat{V}_0(\hat{p}) = - \left[ 1 - \frac{\hat{p}}{2} \ln \frac{1 + \hat{p}}{1 - \hat{p}} \right]^{-1}. \quad (16)$$

Although the detailed form of this potential is specific to our particular choice of cut-off, the fact that it tends to a constant as  $\hat{p} \rightarrow 0$  is a generic feature, which is present for any regulator.

The corresponding unscaled potential is

$$V_0(p, \Lambda) = - \frac{2\pi^2}{M} \left[ \Lambda - \frac{p}{2} \ln \frac{\Lambda + p}{\Lambda - p} \right]^{-1}. \quad (17)$$

The solution to the LS equation for  $K$  with this potential is infinite, or rather  $1/K = 0$ . This corresponds to a system with infinite scattering length, or equivalently a bound state at exactly zero energy.

To study the behaviour near this fixed point we consider small perturbations about it that scale with definite powers of  $\Lambda$ :

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = \hat{V}_0(\hat{p}) + C\Lambda^\nu \phi(\hat{k}', \hat{k}, \hat{p}). \quad (18)$$

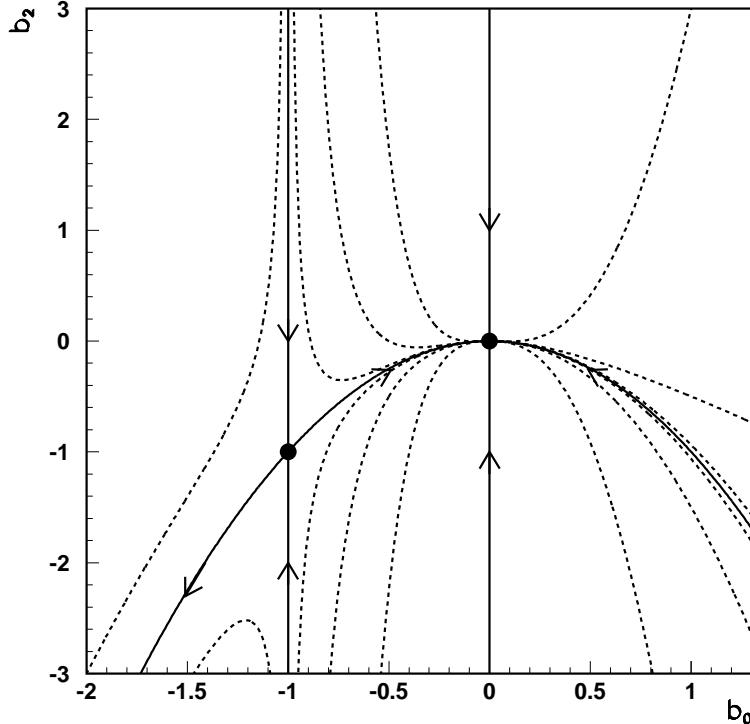
These satisfy the linearised RG equation

$$\hat{k}' \frac{\partial \phi}{\partial \hat{k}'} + \hat{k} \frac{\partial \phi}{\partial \hat{k}} + \hat{p} \frac{\partial \phi}{\partial \hat{p}} + \phi + \frac{\hat{V}_0(\hat{p})}{1 - \hat{p}^2} [\phi(\hat{k}', 1, \hat{p}) + \phi(1, \hat{k}, \hat{p})] = \nu \phi. \quad (19)$$

Solutions to (19) that depend only on energy ( $\hat{p}$ ) can be found straightforwardly by integrating the equation. They are

$$\phi(\hat{p}) = \hat{p}^{\nu+1} \hat{V}_0(\hat{p})^2. \quad (20)$$

Requiring that these be well-behaved as  $\hat{p}^2 \rightarrow 0$ , we find the RG eigenvalues  $\nu = -1, 1, 3, \dots$ . The fixed point is unstable: it has one negative eigenvalue.



**FIGURE 1.** The RG flow of the first two terms in the expansion of the rescaled potential in powers of energy. The two fixed points are indicated by the black dots. The solid lines are flow lines that approach one of the fixed points along a direction corresponding to an RG eigenfunction; the dashed lines are more general flow lines. The arrows indicate the direction of flow as  $\Lambda \rightarrow 0$ .

The instability can be seen from the RG flow in Fig. 1. Only potentials that lie exactly on the “critical surface” flow into the nontrivial fixed point as  $\Lambda \rightarrow 0$ .

Any small perturbation away from this surface eventually builds up and drives the potential either to the trivial fixed point at the origin or to infinity.

The corresponding unscaled potential is

$$V(k', k, p, \Lambda) = V_0(p, \Lambda) + \frac{M}{2\pi^2} (C_{-1} + C_1 p^2 + \dots) V_0(p, \Lambda)^2. \quad (21)$$

For perturbations around the nontrivial fixed point, we can assign an order  $d = \nu - 1 = -2, 0, 2, \dots$  to each term in the potential. This power counting for (energy-dependent) perturbations agrees with that found by Kaplan, Savage and Wise [2] using a “power divergence subtraction” scheme and also by van Kolck [3] in a more general subtractive renormalisation scheme. The equivalence can be seen by making the replacement

$$V_0 = -\frac{2\pi^2}{M\Lambda} + \dots \rightarrow -\frac{4\pi}{M\mu}, \quad (22)$$

where  $\mu$  is the renormalisation scale introduced by Kaplan, Savage and Wise in their subtraction scheme, and which plays an analogous role to the cut-off  $\Lambda$  in our approach.

The on-shell  $K$ -matrix for this potential is (to any order in the  $C$ 's)

$$\frac{1}{K(p, p, p)} = -\frac{M}{2\pi^2} (C_{-1} + C_1 p^2 + \dots). \quad (23)$$

This is just the effective-range expansion (4). There is a one-to-one correspondence between the perturbations in  $V$  and the terms in that expansion,

$$C_{-1} = -\frac{\pi}{2a}, \quad C_1 = \frac{\pi r_e}{4}. \quad (24)$$

The expansion around the nontrivial fixed point is the relevant one for systems with large scattering lengths, such as  $s$ -wave nucleon-nucleon scattering.

## WEAK LONG-RANGE FORCES

The treatment outlined above is only valid at very low momenta, where all pieces of the potential can be regarded as short-range. To extend it to describe nucleon-nucleon scattering at higher momenta, we would like to include pion-exchange forces explicitly. The longest-ranged of these is single pion exchange, which provides a central Yukawa potential,

$$V_{1\pi}(\mathbf{k}', \mathbf{k}) = -\frac{4\pi\alpha_\pi}{(\mathbf{k} - \mathbf{k}')^2 + m_\pi^2}, \quad (25)$$

where

$$\alpha_\pi = \frac{g_A^2 m_\pi^2}{16\pi f_\pi^2} \simeq 0.072. \quad (26)$$

As in chiral perturbation theory, we want to treat the pion mass as a new low-energy scale (in addition to the momentum and energy variables). This can be done by defining a rescaled variable  $\hat{m}_\pi = m_\pi/\Lambda$  and applying the RG as above. The corresponding term in the rescaled potential is

$$\hat{V}_{1\pi}(\hat{\mathbf{k}}', \hat{\mathbf{k}}, \hat{m}_\pi, \Lambda) = -\Lambda \frac{M g_A^2}{8\pi^2 f_\pi^2} \frac{\hat{m}_\pi^2}{(\hat{\mathbf{k}} - \hat{\mathbf{k}}')^2 + \hat{m}_\pi^2}. \quad (27)$$

It scales as  $\Lambda^1$ , like the effective-range term in the potential above. This suggests that one-pion exchange (OPE) can be treated as a perturbation. It would contribute at next-to-leading order (NLO) in the potential.

However questions remain about whether OPE is really weak enough for a perturbative treatment to be useful. A possible scale for nonperturbative long-range physics is the pionic “Bohr radius”:

$$R = \frac{2}{\alpha_\pi M} \simeq 5.8 \text{ fm}. \quad (28)$$

This should be compared with the range of the Yukawa potential,  $r_\pi = 1/m_\pi = 1.4$  fm, which cuts off the potential at long distances, preventing the formation of a bound state. The ratio of these scales is

$$\frac{r_\pi}{R} \simeq 0.24, \quad (29)$$

Although this is smaller than the critical value of 0.84, at which a bound state forms [6], one might expect relatively slow convergence of the perturbation series.

Further questions are raised when the contribution of OPE to the effective range is examined. A perturbative treatment (to NLO in an expansion in powers of momenta,  $m_\pi$  and  $1/a$ , as in [8]) gives a short-range contribution to the effective  ${}^1S_0$  range of

$$r_e^0 = r_e - \frac{2\alpha_\pi M}{m_\pi^2} \quad (30)$$

$$= 2.62 - 1.38 = 1.24 \text{ fm}. \quad (31)$$

It is also possible to set up a distorted-wave effective-range expansion, in which the long-range interaction is treated all orders [9]. This is essentially an expansion in powers of energy of  $p \cot(\delta - \delta_{1\pi})/|\mathcal{F}_{1\pi}(p)|^2$  where  $\delta_{1\pi}$  is the OPE phase shift and  $\mathcal{F}_{1\pi}(p)$  the corresponding Jost function [6]. The resulting purely short-range effective range is [10] (see also [11])

$$r_e^0 = 4.2 \text{ fm}. \quad (32)$$

This is significantly different from the perturbatively corrected effective range (30). The difference may be an indication of either strong forces with two-pion range, or of strong short-range forces with a complicated structure [12].

## SUMMARY

We have applied Wilson's renormalisation group to nonrelativistic two-body scattering and identified two important fixed points [5].

The first is the trivial fixed point. Perturbations around it can be used to describe systems with weak scattering. These perturbations can be organised according to Weinberg's power counting [1].

The second fixed point describes systems with a bound state at exactly zero energy. In this case the relevant power-counting is the one found by Kaplan, Savage and Wise [2] and van Kolck [3]. The expansion around this fixed point is exactly equivalent to the effective-range expansion.

These ideas can be extended in various ways. Short-range interactions in other numbers of spatial dimensions can be studied. The critical dimension for instability of the nontrivial fixed point is  $D = 2$ , which has been studied for some time in the context of anyons [13,14].

Three-body systems are also being studied from the point of view of effective field theory [15,16]. In some cases these display much more complicated behaviour under the RG than the two-body ones discussed above [17].

Various nucleon-nucleon scattering observables as well as deuteron properties have been calculated using the expansion around the nontrivial fixed point [2,18–20]. In this approach, pion-exchange forces are treated as perturbations. An alternative approach which is being explored by other groups is to use Weinberg's power counting in the expansion of the potential, but then to iterate that potential to all orders in the LS equation [21–25]. This may provide a way to evade the problems of slow convergence when OPE is included explicitly [8,12].

Finally, strong long-ranged interactions, such as the Coulomb force, lead to quite different behaviour from the examples discussed here. They can still be treated using similar techniques, as in NRQED [26] and NRQCD [27].

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